Calculus II - Day 8

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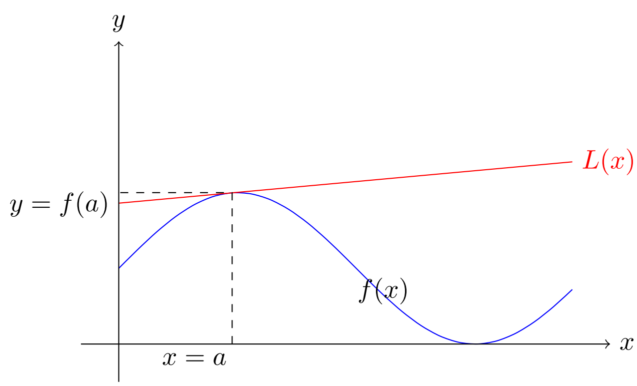
Taylor Polynomials and Approximation

Goals for today:

- Define the degree n Taylor polynomial $p_n(x)$ for a function $f(x)$.
- Use polynomials to estimate function values.
- Bound the error in the estimation using Taylor's Remainder Theorem.

Recall: Linear approximation

Let $f(x)$ be differentiable. We can estimate the value of $f(x)$ near the point $x = a$ using the tangent line:



In this case, for any differentiable function $f(x)$, the linear approximation near $x = a$ is based on the value of the function $f(a)$ and the value of the derivative $f'(a)$ at that point. The equation of the tangent line at $x = a$ is given by:

$$L(x) = f(a) + f'(a)(x - a)$$

<u>Ex.</u> Estimate the value of $\sqrt{4.1}$ using linear approximation.
We are trying to estimate $f(x) = \sqrt{x}$ at $x = 4.1$.

Choose the base point $a = 4$:

$$f(4) = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

The linear approximation is:

$$L(x) = f(4) + f'(4)(x-4) = \left(2 + \frac{1}{4}\right)(x-4)$$

Therefore,

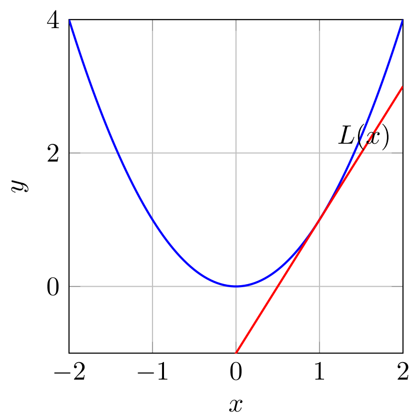
$$\sqrt{4.1} = f(4.1) \approx L(4.1) = 2 + \frac{1}{4} \cdot (4.1 - 4) = 2 + \frac{1}{40} = 2.025$$

Check:

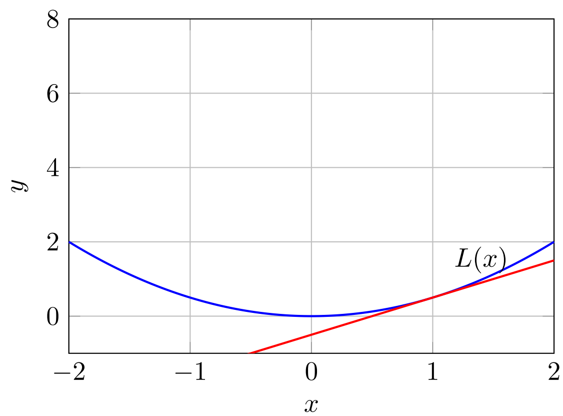
$$(2.025)^2 = 4.100625 \quad \text{(very close!)}$$

This works very well if $f'(x)$ *is not changing very quickly near* $x = a$.

”larger” 2^{nd} derivative - worse approximation



”small” 2^{nd} derivative



To do better: use a higher degree polynomial (quadratic).
Let's approximate $f(x)$ by a quadratic:

$$p_2(x) = c_0 + c_1(x-a) + c_2(x-a)^2$$

How do we choose the constants c_0, c_1, and c_2?

$$f(a) = p_2(a): \quad c_0 = f(a)$$

$$f'(a) = p_2'(a): \quad p_2'(x) = c_1 + 2c_2(x-a)$$

At $x = a$:

$$p_2'(a) = c_1 + 2c_2(a-a) = c_1$$

$\Rightarrow c_1 = f'(a)$

$$p''(a) = p_2''(a): \quad p_2''(a) = 2c_2$$

$$f''(a) = p_2''(a) = 2c_2 \quad \text{(so)} \quad c_2 = \frac{f''(a)}{2}$$

"Quadratic approximation":

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

<u>Ex.</u> For $f(x) = \sqrt{x}$ at $a = 4$:

$$f(a) = \sqrt{4} = 2$$

$$f'(a) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$f''(x) = \frac{d}{dx}\left(\frac{1}{2\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{1}{2}x^{-1/2}\right) = -\frac{1}{4}x^{-3/2}$$

$$f''(a) = -\frac{1}{4} \cdot (4)^{-3/2} = -\frac{1}{4} \cdot \frac{1}{8} = -\frac{1}{32}$$

The quadratic approximation is:

$$\sqrt{x} \approx p_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

Now, estimate $\sqrt{4.1}$:

$$\sqrt{4.1} \approx p_2(4.1) = 2 + \frac{1}{4}(4.1-4) - \frac{1}{64}(4.1-4)^2$$

$$= 2 + \frac{1}{40} - \frac{1}{6400} = 2 + 0.025 - 0.00015625 = 2.0248375$$

$$(2.0248375)^2 = 4.09999221\ldots \quad \text{(better than the linear approximation!)}$$

Definition: Let $f(x)$ be a function that is n times differentiable at $x = a$.
The n-th order Taylor polynomial of f centered at $x = a$ is:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Alternatively, using summation notation:

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

where $0! = 1$ and $f^{(0)}(x) = f(x)$.

Finding a Taylor polynomial requires us to compute derivatives quickly.
Ex. Find the degree 3 Taylor polynomial of $f(x) = e^x$ centered at $x = 0$.
(This is a Maclaurin polynomial: $a = 0$)
Need $f^{(k)}(0)$ for $k = 0, 1, 2, 3$.

k	$f^{(k)}(x)$	$f^{(k)}(0)$	$c_k = \frac{f^{(k)}(0)}{k!}$
0	e^x	1	$\frac{1}{0!} = \frac{1}{1} = 1$
1	e^x	1	$\frac{1}{1!} = 1$
2	e^x	1	$\frac{1}{2!} = \frac{1}{2}$
3	e^x	1	$\frac{1}{3!} = \frac{1}{6}$

$f_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$
When x is near 0, $p_3(x) \approx e^x$: use this to estimate $\sqrt{e}$.

$$e^{1/2} \approx p_3(1/2) = 1 + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{6}\left(\frac{1}{2}\right)^3 = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48}$$

$$= \frac{48 + 24 + 6 + 1}{48} = \frac{79}{48}$$

Ex. $f(x) = \cos(x)$. Find the degree 5 Taylor polynomial at $x = 0$.
Need $f^{(k)}(0)$ for $k = 0, 1, 2, 3, 4, 5$.

k	$f^{(k)}(x)$	$f^{(k)}(0)$	$c_k = \frac{f^{(k)}(0)}{k!}$
0	$\cos(x)$	1	1
1	$-\sin(x)$	0	0
2	$-\cos(x)$	-1	$\frac{-1}{2!} = \frac{-1}{2}$
3	$\sin(x)$	0	0
4	$\cos(x)$	1	$\frac{1}{4!} = \frac{1}{24}$
5	$-\sin(x)$	0	0

$p_5(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

Ex. Find the degree 4 Taylor polynomial of $\ln(x)$ centered at $x = 1$.
Need $f^{(k)}(1)$ for $k = 0, 1, 2, 3, 4$.

k	$f^{(k)}(x)$	$f^{(k)}(1)$	$\frac{f^{(k)}(1)}{k!}$
0	$\ln(x)$	$\ln(1) = 0$	0
1	$\frac{1}{x} = x^{-1}$	1	1
2	$-x^{-2}$	-1	$\frac{-1}{2!} = \frac{-1}{2}$
3	$2x^{-3}$	2	$\frac{2}{3!} = \frac{1}{3}$
4	$-6x^{-4}$	-6	$\frac{-6}{4!} = \frac{-1}{4}$

The degree 4 Taylor polynomial for $\ln(x)$ centered at $x = 1$ is:

$$p_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

Q: How good an approximation is $p_n(x)$ to $f(x)$?
Definition: Let $p_n(x)$ be the degree n Taylor polynomial of $f(x)$ centered at $x = a$. When we use $p_n(x)$ to estimate $f(x)$, the remainder is:

$$R_n(x) = f(x) - p_n(x)$$

Taylor's Remainder Theorem

Suppose the first $n+1$ derivatives of $f(x)$ are continuous on the interval from x to a (either $[x, a]$ or $[a, x]$). For all x in this interval, if

$$f(x) = p_n(x) + R_n(x),$$

then

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between x and a.
How do we use this?
Find the maximum value M of $|f^{(n+1)}(c)|$ for c between x and a. Then:

$$|R_n(x)| \leq M \cdot \frac{|x-a|^{n+1}}{(n+1)!}$$

Ex. The degree 5 Taylor polynomial for $\cos(x)$ at $a = 0$ is:

$$p_5(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

How far is $p_5(-0.1)$ from the actual value of $\cos(-0.1)$?
We need to find an upper bound on the 6th derivative of $\cos(x)$ between $[-0.1, 0]$.

$$\frac{d^6}{dx^6}\cos(x) = -\cos(x)$$

Since $|\cos(x)|$ is always between 0 and 1, take $M = 1$.

$$|R_5(-0.1)| \leq M \cdot \frac{|x-a|^{n+1}}{(n+1)!}, \quad n = 5,\, M = 1,\, a = 0,\, x = -0.1$$

$$= 1 \cdot \frac{|0.1|^6}{6!} = \frac{0.1^6}{720} = \frac{1}{720000000}$$

Ex. The degree 3 Taylor polynomial of e^x is:

$$p_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

How far is $p_3\left(\frac{1}{2}\right)$ from $\sqrt{e}$?

We need an upper bound on the 4th derivative of e^x between 0 and $\frac{1}{2}$.

$$\frac{d^4}{dx^4} e^x = e^x$$

Take $M = \sqrt{e}$.

Why? e^x is increasing, so its maximum is attained at the right endpoint $x = \frac{1}{2}$.

$$|R_3\left(\frac{1}{2}\right)| \leq \sqrt{e} \cdot \frac{|0.5|^4}{4!} = \frac{\sqrt{e}}{384}$$